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ADDENDUM

Exact thermodynamics of a planar array of Ginzburg–Landau chains with next-nearest-neighbour interaction

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Abstract
An exact formula for the free energy of a planar array of ψ^4 chains with next-nearest-neighbour (nnn) interaction is obtained, and the analogy with the Ising model is discussed.

In a previous paper [1], we proposed a simple method allowing, in principle, the exact evaluation of the free energy of a planar array of Ginzburg–Landau chains with nearest-neighbour (nn) inter-chain interaction. This method can be extended to 3D [2] and, in 2D, to the case when the next-nearest-neighbour (nnn) inter-chain interactions are taken into account. We shall discuss here this second issue. A more detailed variant of this discussion can be found in [3].

So, we shall consider a planar array of Ginzburg–Landau chains, described by the functional

$$\mathcal{F}_{\mathrm{GL}}[\psi] = \sum_{j=1}^{N} \int_0^L \frac{\mathrm{d}x}{\xi_0} \left[a\psi_j^2 + b\psi_j^4 + c\left(\frac{\mathrm{d}\psi_j}{\mathrm{d}x}\right)^2 + c_1\left(\psi_{j+1} - \psi_j\right)^2 + c_2\left(\psi_{j+2} - \psi_j\right)^2 \right] \tag{1}$$

where the field $\psi_j(x)$, on the jth chain, is real and satisfies cyclic boundary conditions. The parameters a, b, c, ξ_0 have the same meaning as in [1]; c_1 corresponds to $c_\perp$ of [1]. The transfer matrix Hamiltonian, associated with (1), is

$$H_{\mathrm{TM}} = \sum_{j=1}^{N} \left[-\frac{1}{2m}\frac{\partial^2}{\partial\psi_j^2} + a\psi_j^2 + b\psi_j^4 + c_1\left(\psi_{j+1} - \psi_j\right)^2 + c_2\left(\psi_{j+2} - \psi_j\right)^2 \right]. \tag{2}$$

Following closely the approach described in [1], we obtain the free energy of the system:

$$F = N\Omega_0 \frac{k_{\mathrm{B}} T \tau_n}{2\pi} \sqrt{1 + \frac{2\left(c_1 + c_2\right)}{a'\tau_n \Omega_0^2}}\, E\left(k_1, k_2\right) \tag{3}$$

where $\tau_n = t - t_n$, $t_n = 1 - \frac{2}{a'}(c_1 + c_2)$ and

$$E\left(k_1, k_2\right) = \int_0^{\frac{\pi}{2}} \sqrt{\left(1 - k_1^2 \sin^2\theta\right)\left(1 - k_2^2 \sin^2\theta\right)}\, \mathrm{d}\theta. \tag{4}$$

k_1^2, k_2^2 are parameters depending on c_1, c_2. If $c_1 > 0, c_2 < 0, c_1 > |c_2|$, they satisfy the conditions

$$k_1^2 > k_2^2 > 0; \qquad k_2^2 \to 0 \qquad \text{if } c_2 \to 0. \tag{5}$$

The function $E(k_1, k_2)$ can be considered a generalization of the complete integral of the second kind, $E(k)$. For $k_2 = 0$, corresponding to the case when the nnn interaction is neglected, $E(k_1, 0) = E(k_1)$, and (3) reduces to our previous result, equation (69) of [1]. In the general case, using the formulae (252.19) and (262.17) of [4], $E(k_1, k_2)$ can be put in the form

$$E\left(k_1, k_2\right) = -\frac{1}{2\alpha^2\left(\alpha^2 - 1\right)} \frac{1}{\sqrt{1 - k_2^2}} \left[\alpha^2 E + \left(k^2 - \alpha^2\right) K + \left(2\alpha^2 - \alpha^4 - k^2\right) \Pi\left(\alpha^2, k\right)\right] \tag{6}$$

where

$$\alpha^2 = -\frac{k_2^2}{1 - k_2^2} < 0, \qquad k^2 = \frac{k_1^2 - k_2^2}{1 - k_2^2} \tag{7}$$

and the conventions of [4] for elliptic integrals and functions have been adopted.

The formulae (3) and (6) give the exact expression for the free energy of our system, (1). Let us briefly discuss the critical

behaviour of the system. According to (3) and (7), the free energy is proportional to

$$-k_2^2\left(1-k_2^2\right)E(k)+k_1^2\left(1-k_2^2\right)K(k)$$
$$+\left(k_1^2k_2^2-k_1^2-k_2^2\right)\Pi(\alpha^2,k). \tag{8}$$

The singularities of this expression might appear at $k=1$, due to $K(k)$ and $\Pi(\alpha^2,k)$ terms. In fact, with (410.01) and (904.01) of [4], we can see that these singularities compensate each other, and, for $k^2\to 1$, the most singular contribution is proportional to

$$k'^2\ln\frac{1}{k'}. \tag{9}$$

This cancellation is quite similar to that obtained by Fan and Wu [5] in their calculation of the specific heat of a 2D Ising model with nnn interaction. The term (9) gives a logarithmic singularity of the specific heat, which occurs at a critical temperature given by the equation $k=1$ or, equivalently (see equation (8)), $k_1^2=1$. So, the presence of a nnn term does not change qualitatively the critical behaviour of the system.

The influence of the nnn interaction on the critical behaviour for the Ising model was a subject of intensive debate (as is quite generally accepted, the planar array of Ginzburg–Landau chains belongs to the 2D Ising universality class). The conclusion of several analytical approximations [5, 6] and Monte Carlo simulations [7, 8] is that the nn Ising critical behaviour is not qualitatively modified by the nnn interaction, at least for small values of the coupling constant c_2.

Our result agrees with this conclusion. As (3) is also the ground state energy of the chain of quantum anharmonic oscillators coupled via nn and nnn interactions, described by the transfer matrix Hamiltonian (2), the same result can be used in the study of quantum phase transitions in this system.

References

[1] Bârsan V 2006 *J. Phys.: Condens. Matter* **18** 9273
[2] Bârsan V 2007 *Preprint* 0708.2325v1 [math-ph]
[3] Bârsan V 2007 *Preprint* 0710.0219v1 [cond-mat.stat-mech]
[4] Byrd P F and Friedman M D 1971 *Handbook of Elliptic Integrals for Engineers and Scientists* 2nd edn (Berlin: Springer)
[5] Fan C and Wu F Y 1969 *Phys. Rev.* **179** 560
[6] Grynberg M D and Tanatar B 1992 *Phys. Rev.* B **45** 2876
[7] Landau D P 1980 *Phys. Rev.* B **21** 1285
[8] Aguilera-Granja F and Moran-Lopez J L 1993 *J. Phys.: Condens. Matter* **5** A195